Title: Iterating Linear Functions

Brief Overview:

Students will use a spreadsheet to study the sequences of real numbers obtained by iteration of linear functions. They will discover the various types of behavior that can result--divergence, convergence to a fixed point, and alternating behavior--and when it will occur. They will use algebra to prove their conjectures.

Links to NCTM 2000 Standards:

- **Mathematics as Problem Solving**
  Students will solve mathematical problems by using spreadsheets to look for patterns, make conjectures, and prove them algebraically.

- **Mathematics as Reasoning and Proof**
  Students will use inductive reasoning to discover how the slope and y-intercept affect the behavior of the function under iteration. Then they will use deductive reasoning to prove their findings algebraically.

- **Mathematics as Communication**
  Students will explain relevant terms and concepts, as well as their findings, both orally and in writing.

- **Mathematics as Connections**
  This topic connects algebra, discrete mathematics, and analysis.

- **Mathematics as Representation**
  Students will use time-series graphs to picture the behavior of linear functions under iteration.

- **Patterns, Functions, and Algebra**
  Students will investigate functions from the perspective of iteration. They will study sequences generated by iteration and find their limits if they exist. They will solve equations and work with algebraic expressions to prove their conjectures.

Links to Maryland High School Mathematics Core Learning Goals:

Functions and Algebra

- **1.1.2**
  Students will find the expression for the fixed points of linear functions in terms of their slope and y-intercept.

Geometry, Measurement, and Reasoning

- **2.2.3**
  Students will use inductive reasoning to discover how the slope and y-intercept affect the behavior of the function under iteration. Then they will use deductive reasoning to prove their findings algebraically.

Grade/Level:

Grades 9-12

Duration/Length:

This activity will take 4 or 5 days, depending on class duration and students' prior knowledge.

Prerequisite Knowledge:

Students should have working knowledge of the following concepts:

- Graphing in the coordinate plane
- Functions and function notation
- Slope and y-intercept of linear functions
- Solving linear equations
- Behavior of infinite geometric series (in order to prove one of the findings)

Student Outcomes:

Students will:

- use a spreadsheet to investigate the orbits of linear functions under iteration.
- make time-series graphs to represent the orbits and determine their behavior.
- use inductive reasoning to make conjectures about the convergence and divergence of the resulting orbits.
- use algebra to prove those conjectures.

Materials/Resources/Printed Materials:

- Graph paper
- Computers
- Spreadsheet software

Development/Procedures:

Prepare a spreadsheet file ahead of time as follows:

	A	B
1	m =	
2	b =	
3		
4	Iteration no.	Orbit
5	0	0
6	=a5+1	=b$1*b5+b$2

Copy row 6 to rows 7 through 35. This will calculate the first 30 iterations of the function $f(x) = mx + b$, where the values of m and b are entered by the user in cells B1 and B2, respectively. Create a time-series graph of these iterates by setting the x-series to cells A5 through A35 and the y-series to cells B5 through B35. The specific commands to do this vary depending on the software used, so check your documentation if necessary. Position the graph so it is visible on the screen to the right of column B. Make this spreadsheet available on each computer your students will use.

Define iteration: Given a function f and an **initial point** x_0, form a sequence as follows: Let $x_1 = f(x_0)$, $x_2 = f(x_1)$, …, $x_{n+1} = f(x_n)$. This process is called **iteration**, and the resulting sequence is called the **orbit** of x_0.

As an example, consider the function $f(x) = 0.5x + 1$. Have students work in pairs to find the first ten terms of the orbit, assigning a different initial point (e.g., $x_0 = -2, -1, 0, 1, 2$) to each pair. All students should discover that the orbits seem to converge to 2.

Have students evaluate $f(2)$, noting that $f(2) = 2$.

Point out that 2 is called a fixed point for $f(x) = 0.5x + 1$. Define a **fixed point** as a number that solves the equation $f(x) = x$. In addition, 2 is an **attracting** fixed point (or **attractor**) because orbits converge to it; that is, terms of the orbit get arbitrarily close to the fixed point.

Show students how to make a **time-series graph** by plotting the values x_n of the orbit against the iteration numbers n.

Now assign each pair of students a different linear function with slope between 0 and 1, and an integral y-intercept. Using values of 0.5, 0.25, 0.75, 0.4, and 0.8 for the slope will help students more easily discover the pattern later on. With $x_0 = 0$, have them make a time-series graph and determine the value of the attractor if it exists.

Next show students how to do this on the spreadsheet. Have them enter their particular values of m and b into cells B1 and B2, and leave 0 in cell B5 for x_0. The orbit that appears in column B should agree with their calculations, and the time-series graph plotted on the computer should agree with the one they did by hand.

Now have each pair keep their same values of m and b, but try several different initial points in cell B5. Students should find that regardless of the initial point, the orbit always converges to a certain fixed point that is different for each function. Make a list of the results on the chalkboard in the form of a table with three columns: m, b, and the attractor. (The attractor should be $b/(1 - m)$, but do not tell students this yet.)

Next, repeat the above process with a slope between 0 and -1. Students should notice that when the slope is negative, the orbit alternately increases and decreases, but still converges to a certain fixed point, regardless of the initial point. (Again, the attractor is $b/(1 - m)$.)

Discuss the results found so far. Lead students to notice that for $m = 0.5$, the attractor is twice the y-intercept. Have them verify that these values are indeed fixed points. Ask them for conjectures for other values of m. For example, when $m = 0.75$, the attractor is $4b$, and when $m = 0.25$, the attractor is $4b/3$. Challenge them to find similar expressions for the attractors for specific values of m (both greater than and less than 0) or for the general case.

Have students solve the equation $f(x) = x$ (that is, $mx + b = x$) for x to find the general expression for the fixed point. They should obtain $x = b/(1 - m)$. Have them verify that this agrees with previous conjectures and with their experimental results. Emphasize that this only proves that the numbers so found are fixed points, not that they are attractors. The next part of the investigation demonstrates why it is important to make this distinction.

Now assign to each pair of students a linear function with a slope whose absolute value is greater than one. Without using the spreadsheet, have them predict the fixed point (they should use the equation $x = b/(1 - m)$), and check it in the function by hand. Then have students enter their values of m and b into the spreadsheet and look at the orbits. They should see that the orbits do not converge, even though fixed points do exist.

Ask students what is different about these functions. (Their slopes are greater than 1 in absolute value.) Also ask them what happens when the slope is negative. (The orbits alternately increase and decrease, as before.)

Thus, $x = b/(1 - m)$ always yields the fixed point, but the fixed point seems to be an attractor only when $|m| < 1$.

Lead students through the following proof that if p is a fixed point, then when $|m| < 1$, all orbits generated by iterating $f(x) = mx + b$ converge to p. Also, when $|m| > 1$, all orbits diverge. We will prove this by showing that the differences $|x_n - p|$ as n approaches infinity converge to 0 when $|m| < 1$, and diverge when $|m| > 1$.

Since $x_{n+1} = f(x_n)$ and $f(p) = p$,

$$|x_{n+1} - p| / |x_n - p| = |f(x_n) - f(p)| / |x_n - p|$$
$$|x_{n+1} - p| / |x_n - p| = |m|$$
$$|x_{n+1} - p| = |m|\,|x_n - p|$$

In other words,

$$|x_1 - p| = |m|\,|x_0 - p|$$
$$|x_2 - p| = |m|\,|x_1 - p| = |m|^2\,|x_0 - p|$$
$$|x_3 - p| = |m|\,|x_2 - p| = |m|^3\,|x_0 - p|$$

In general,

$$|x_n - p| = |m|\,|x_0 - p| = |m|^n\,|x_0 - p|$$

Since m, x_0, and p are fixed as n approaches infinity, the right side of the equation is a geometric sequence in n with constant ratio $|m|$. Therefore, when $|m| < 1$, the right side converges to 0, and so x_n converges to p. When $|m| > 1$, the right side diverges so x_n also diverges.

Assessment:

Have students write a paper describing their investigation and findings. They should describe the role of m, b, and x_0 in determining the orbit. The should explain what a fixed point is, how a time-series graph is made, what kind of behavior results, and when it results. They should explain, using correct algebra, why such behavior results.

A holistic scoring rubric may be used to evaluate aspects of the students' writing:
1. Description of mathematical concepts involved
2. Explanation of procedures used
3. Description of findings
4. Use of notation and algebraic manipulation
5. Use of logical reasoning

Each of these aspects can be scored on the following scale:

4	Work is correct and complete
3	Work is almost correct and complete; some errors were made or details omitted.
2	Work shows a general understanding, but notable gaps or errors are present.
1	Some work is correct, showing minimal or incomplete understanding, but there is little or no chain of reasoning.
0	Work is incorrect or meaningless. There is no evidence of understanding.

Extension/Follow Up:

Students may wish to apply the above procedures to non-linear functions. The simplest would be quadratic functions. For those who know or can learn calculus, results can be explained by examining the derivative instead of slope.

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